JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **33**, No. 1, February 2020
http://dx.doi.org/10.14403/jcms.2020.33.1.173

DYNAMICAL PROPERTIES ON ITERATED FUNCTION SYSTEMS

Hahng-Yun Chu, Minhee Gu, Se-Hyun Ku∗, and Jong-Suh Park

Abstract. Let X be a compact space and Λ a finite index set. We deal with dynamical properties of iterated function systems on X. For an iterated function system $\mathcal{F}$ on X, we prove that $\mathcal{F}$ is c-expansive if and only if $\mathcal{F}^k$ is also c-expansive for each $k \in \mathbb{N}$. Furthermore we prove that the c-expansiveness of $\mathcal{F}$ is equivalent to the original expansiveness of the shift map of it.

1. Introduction

Let (X, d) be a compact metric space and Λ a nonempty finite set. We first consider an *Iterated Function System*(shortly, IFS) on X under Λ. For every $\lambda \in \Lambda$, let f_λ be a continuous surjection from X to itself. The IFS $\mathcal{F}$ is defined by

$$\mathcal{F} := (X, \{f_\lambda : X \to X \mid \lambda \in \Lambda\})$$

which is a family of continuous surjective maps $\{f_\lambda : X \to X \mid \lambda \in \Lambda\}$ under the composition. Especially, when one studies the field of fractal geometry and fractal dynamical systems, a lot of theories about IFS are much valuable. From Hutchinson [6] introduced the current form of IFSs, many scholars studied this concept related to the dynamic notions of attractors and shadowing properties [2, 7, 8].

For the study of expansiveness of an invertible iterated function system(shortly, IIFS), Chu *et al.* investigated the dynamics of the systems

---

Received February 11, 2020; Accepted February 14, 2020.

2010 Mathematics Subject Classification: Primary 54H20, Secondary 37B25.

Key words and phrases: iterated function systems, expansive, c-expansive.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2018R1D1A1B07051).

∗ Corresponding author.

on compact spaces. They found the equivalences for the notions of expansiveness on the invertible iterated function systems. (see [4, 5]).

In this paper, we introduce the notion of c-expansiveness of an IFS and investigate the properties of the notion. We show that an IFS $\mathcal{F}$ is c-expansive if and only if each iteration of $\mathcal{F}$ is also c-expansive. We also show that the c-expansiveness of $\mathcal{F}$ is equivalent to the original expansiveness of the shift map of it.

From now on, let (X, d) be a compact metric space and Λ a nonempty finite set.

2. c-expansiveness on IFS

In this section, we first present an expansive homeomorphism on the compact space. A homeomorphism $f : X \to X$ is *expansive* if there exists a positive constant e such that $x \neq y (x, y \in X)$ implies

$$d(f^n(x), f^n(y)) > e$$

for some integer n. We call the constant e the *expansive constant* for f. For a compact space X, we canonically consider the product topological space

$$X^{\mathbb{Z}} := \{(x_i) \mid x_i \in X,\ i \in \mathbb{Z}\}.$$

So $X^{\mathbb{Z}}$ is a compact space. For $(x_i), (y_i) \in X^{\mathbb{Z}}$, we define a compatible metric $\widetilde{d}$ for the space $X^{\mathbb{Z}}$ given by

$$\widetilde{d}((x_i), (y_i)) := \sum_{i=-\infty}^{\infty} \frac{d(x_i, y_i)}{2^{|i|}}.$$

For the details of the definitions, see [1, 3].

Let $\mathcal{F}$ be an IFS on X which consists of the family of continuous surjections on X under the composition, i.e.,

$$\mathcal{F} := (X, \{f_\lambda : X \to X \mid f_\lambda : \text{continuous surjection for each } \lambda \in \Lambda\}).$$

Put

$$X_{\mathcal{F}} := \{(x_i) \in X^{\mathbb{Z}} \mid \text{for every } i \in \mathbb{Z},\ f_{\lambda_i}(x_i) = x_{i+1} \text{ for some } \lambda_i \in \Lambda\}.$$

Define a shift map σ on $X_{\mathcal{F}}$ to itself such that for $(x_i) \in X_{\mathcal{F}}$ and $j \in \mathbb{Z}$, $(\sigma(x_i))_j = x_{j+1}$. For $(x_i) \in X_{\mathcal{F}}$, we also take a sequence (f_{λ_i}) of mappings satisfying the property that $f_{\lambda_i}(x_i) = x_{i+1}$. We consider a restricted metric $\tilde{d}$ on $X_{\mathcal{F}}$ given by $\tilde{d}((x_i), (y_i)) := \sum_{i=-\infty}^{\infty} d(x_i, y_i)/2^{|i|}$ for $(x_i), (y_i) \in X_{\mathcal{F}}$.

The IFS $\mathcal{F}$ is *c-expansive* if there exists a positive constant e such that $(x_i), (y_i) \in X_{\mathcal{F}}$ with $d(x_i, y_i) \le e$ for all $i \in \mathbb{Z}$ implies $(x_i) = (y_i)$, that is, $(x_i) \neq (y_i)$ implies $d(x_i, y_i) > e$ for some $i \in \mathbb{Z}$. Here, e is called an *expansive constant* for $\mathcal{F}$. We called the IFS $\mathcal{F}$ a c-expansive IFS with an expansive constant e.

For $k \ge 1$, put

$$\mathcal{F}^k = (X, \{f_{\lambda_k} \circ \cdots \circ f_{\lambda_1} : X \to X \mid \lambda_1, \cdots, \lambda_k \in \Lambda\}).$$

So we similarly have $X_{\mathcal{F}^k}$ obtained by

$$\{(x_{ki}) \in X^{\mathbb{Z}} \mid \forall i \in \mathbb{Z},\ f_{\lambda_{i_k}} \circ \cdots \circ f_{\lambda_{i_1}}(x_{ki}) = x_{k(i+1)} \text{ for some } \lambda_{i_j} \in \Lambda\}.$$

Now we mention briefly IIFSs and notions of expansiveness on the systems. Let $\mathcal{F}^+ := (X, \{f_\lambda : X \to X \mid \lambda \in \Lambda^+\})$ be an IFS. Assume that every continuous mapping f_λ in $\mathcal{F}^+$ is a homeomorphism from X to itself. We denote an index λ^{-1} satisfying the property that $f_\lambda \circ f_{\lambda^{-1}} = f_{\lambda^{-1}} \circ f_\lambda = id_X$. Additionally we assume that

$$\mathcal{F}^+ = (X, \{f_\lambda : X \to X \mid \lambda \in \Lambda^+\})$$

be an IFS consists of homeomorphisms from X to itself such that $f_{\lambda_1} \circ f_{\lambda_2} \neq id_X$ for all λ_1 and $\lambda_2 \in \Lambda^+$. We define an *invertible IFS* (in short, IIFS) $\mathcal{F}$ from the IFS $\mathcal{F}^+$ given by $\mathcal{F} := (X, \{f_\lambda,\ f_{\lambda^{-1}} \mid \lambda \in \Lambda^+\})$. We denote $\Lambda^- := \{\lambda^{-1} | \lambda \in \Lambda^+\}$ and $\Lambda := \Lambda^+ \cup \Lambda^-$. Then we get that $\mathcal{F} = (X, \{f_\lambda \mid \lambda \in \Lambda\})$.

An IIFS $\mathcal{F}$ is *expansive* if there exists a positive constant e such that for each $x, y \in X$ with $x \neq y$, there exist a positive integer n and $(\lambda_1, \cdots, \lambda_n) \in \Lambda^n$ satisfying

$$d(f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(x), f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(y)) > e.$$

Here, e is called an *expansive constant* for $\mathcal{F}$.

An IIFS $\mathcal{F}$ is *rigidly expansive* provided that if there exists a sequence $\sigma = \{\lambda_i\}_{i=1}^{\infty} \in \Lambda^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$,

$$d(f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(x), f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(y)) \le e$$

then $x = y$. Here, e is called a *rigidly expansive constant* for $\mathcal{F}$.

Remark 2.1. Let $\mathcal{F}$ be an IIFS. Then we have that $\mathcal{F}$ is rigidly expansive if and only if $\mathcal{F}$ is c-expansive.

Using the above remark and Corollay 2.4 in [4], we can restate for the equivalences on the IIFS.

Proposition 2.2. *Let (X, d) be a compact metric space. Let $\mathcal{F} = (X, \{f_\lambda : X \to X \mid \lambda \in \Lambda\})$ be an IIFS such that for every $\lambda \in \Lambda$, $f_\lambda : X \to X$ is a homeomorphism. Then the following statements are equivalent.*

(1) *$\mathcal{F}$ is c-expansive.*
(2) *$\mathcal{F}$ is rigidly expansive.*
(3) *$\mathcal{F}$ has a rigid generator.*
(4) *$\mathcal{F}$ has a weak rigid generator.*

By Corollay 2.6 in [4], we know that the existence of the rigid generators of the IIFS is equivalent to that of the rigid generators of iterations of the IIFS. Combining Proposition 2.2 and the above statement, we direclty the following remark.

Remark 2.3. Let $\mathcal{F} = (X, \{f_\lambda : X \to X \mid \lambda \in \Lambda\})$ be an IIFS on a compact metric space X. For $k \geq 2$, $\mathcal{F}$ is c-expansive if and only if so is $\mathcal{F}^k$.

Now we deal with an IFS $\mathcal{F}$ which consists of the family of continuous surjections on the compact space X with respect to the composition.

Theorem 2.4. *Let $\mathcal{F} = (X, \{f_\lambda : X \to X \mid \lambda \in \Lambda\})$ be an IFS on a compact metric space X and $k \in \mathbb{N}$. Then $\mathcal{F}$ is c-expansive if and only if $\mathcal{F}^k$ is also c-expansive.*

Proof. We first deal with the "only if " condition. Assume that $\mathcal{F}$ is c-expansive with an expansive constant e. Note that for every $\lambda \in \Lambda$, a function f_λ is uniformly continuous. From the uniform continuity, each finite composition of the functions of the form f_λ is also uniformly continuous. So there exists a positive constant e' such that $d(x, y) < e'$ implies for every $1 \leq n \leq k$,

$$d(f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \cdots \circ f_{\lambda_1}(x),\ f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \cdots \circ f_{\lambda_1}(y)) < e,$$

for every $f_{\lambda_i} \in \mathcal{F}$, $i \in \{1, 2, \cdots, n\}$. Thus the constant e' is an expansive constant for $\mathcal{F}^k$. Indeed, we let $(x_{ki}), (y_{ki}) \in X_{\mathcal{F}^k}$ with $d(x_{ki}, y_{ki}) \leq e$. Then for every $i \in \mathbb{Z}$, there exist $\lambda_{i_j} \in \Lambda$ $(j = 1, \cdots, k)$ such that

$$f_{i_k} \circ \cdots \circ f_{i_1}(x_{ki}) = x_{k(i+1)}.$$

For $i \in \mathbb{Z}$, we denote $x_{ki+j} := f_{i_j} \circ \cdots \circ f_{i_1}(x_{ki})$ for all $j \in \{1, \cdots, k\}$. From the uniform continuities in the family, we get that

$$d(x_n, y_n) < e$$

for all $n \in \mathbb{Z}$. By the assumption, we have $(x_i) = (y_i)$. It follows that $(x_{ni}) = (y_{ni})$ for all $i \in \mathbb{Z}$.

Conversely, the " if " condition of this proof is clear.

□

THEOREM 2.5. *Let $\mathcal{F} = (X, \{f_\lambda : X \to X \mid \lambda \in \Lambda\})$ be an IFS on the compact metric space X. Then $\mathcal{F}$ is c-expansive if and only if $\sigma : X_{\mathcal{F}} \to X_{\mathcal{F}}$ is expansive.*

Proof. Assume that an IFS $\mathcal{F}$ is c-expansive with an expansive constant e. Let (x_i) and (y_i) are elements of $X_{\mathcal{F}}$ with $(x_i) \neq (y_i)$. By the c-expansivity of $\mathcal{F}$ there exists an integer n_0 such that

$$d(x_{n_0}, y_{n_0}) > e$$

and hence $\tilde{d}(\sigma^{n_0}((x_i)), \sigma^{n_0}((y_i))) \geq d(x_{n_0}, y_{n_0}) > e$. Therefore σ is expansive with the expansive constant e.

Conversely, let σ be an expansive shift map with an expansive constant e. Let (x_i) and (y_i) be elements of $X_{\mathcal{F}}$ such that $d(x_i, y_i) < e/4$ for every $i \in \mathbb{Z}$. Then $\frac{e}{4}$ is an expansive constant for $\mathcal{F}$. Indeed, we get that

$$\begin{aligned}
\tilde{d}(\sigma^n((x_i)), \sigma^n((y_i))) &= \sum_{i=-\infty}^{\infty} \frac{d(x_{i+n}, y_{i+n})}{2^{|i|}} \\
&= \sum_{i=-\infty}^{-1} \frac{d(x_{i+n}, y_{i+n})}{2^{|i|}} + d(x_n, y_n) + \sum_{i=1}^{\infty} \frac{d(x_{i+n}, y_{i+n})}{2^{|i|}} \\
&= \sum_{i=1}^{\infty} \frac{d(x_{-i+n}, y_{-i+n})}{2^{i}} + d(x_n, y_n) + \sum_{i=1}^{\infty} \frac{d(x_{i+n}, y_{i+n})}{2^{i}} \\
&< e
\end{aligned}$$

for every $n \in \mathbb{Z}$. By the expansiveness of the shift map, we obtain $(x_i) = (y_i)$. Hence $\mathcal{F}$ is c-expansive with the expansive constant $e/4$ which completes this proof.

□

Using the proofs of the above theorems, we directly obtain the following statements related to the expansive properties of IFSs.

Remark 2.6. 1. Let $\mathcal{F} = (X, \{f_\lambda : X \to X \mid \lambda \in \Lambda\})$ be a c-expansive IFS on the compact metric space X and Y a closed subset of X such that $f_\lambda(Y) = Y$ for all $\lambda \in \Lambda$. Then we have $\mathcal{F}|_Y := (Y, \{f_\lambda|_Y : Y \to Y \mid \lambda \in \Lambda\})$ is also c-expansive.

2. Let (X, d_X) and (Y, d_Y) be compact metric spaces. Assume that $\mathcal{F} = (X, \{f_\lambda : X \to X \mid \lambda \in \Lambda\})$ and $\mathcal{G} = (Y, \{g_\gamma : Y \to Y \mid \gamma \in \Gamma\})$ are c-expansive. Then the product IFS $\mathcal{F} \times \mathcal{G} = (X \times Y, \{f_\lambda \times g_\gamma : X \times Y \to X \times Y \mid \lambda \in \Lambda,\ \gamma \in \Gamma\})$ is also c-expansive. Furthermore, every finite product of c-expansive IFSs is also c-expansive.

References

[1] N. Aoki and K. Hiraide, *Topological theory of dynamical systems,* North-Holland, 1994.

[2] M. F. Barnsley and A. Vince, *The Conley attractor of an iterated function system*, Bull. Aust. Math. Soc., **88** (2013), 267–279.

[3] B. F. Bryant, *Expansive self-homeomorphisms of a compact metric space*, Amer. Math. Monthly, **69** (1962), 386–391.

[4] H.-Y. Chu, M. Gu, and S.-H. Ku, *Expansiveness on invertible iterated Function Systems,* J. Chungcheong Math. Soc., **32** (2019), 529–535.

[5] H.-Y. Chu, M. Gu, and S.-H. Ku, *A note on measure expansive iterated function systems,* in preperation

[6] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J., **30** (1981), 713–747.

[7] M. F. Nia, *Iterated function systems with the average shadowing property*, Topology Proc., **48** (2016), 261–275.

[8] M. F. Nia and F. Rezaei, *Hartman-Grobman theorem for iterated function systems*, Rocky Mountain J. Math., **49** (2019), 307-333

Department of Mathematics,
Chungnam National University,
99 Daehak-ro, Yuseong-gu, Daejeon 34134,
Republic of Korea
E-mail: hychu@cnu.ac.kr

Department of Mathematics,
Chungnam National University,
99 Daehak-ro, Yuseong-gu, Daejeon 34134,
Republic of Korea
E-mail: minheegu@cnu.ac.kr

*
Department of Mathematics,
Chungnam National University,
99 Daehak-ro, Yuseong-gu, Daejeon 34134,
Republic of Korea
E-mail: shku@cnu.ac.kr

Department of Mathematics,
Chungnam National University,
99 Daehak-ro, Yuseong-gu, Daejeon 34134,
Republic of Korea
E-mail: jpark@cnu.ac.kr